# Dyson–Schwinger studies of meson masses and decay constants

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The masses and decay constants of the light mesons are studied within a ladder-rainbow truncation of the set of Dyson–Schwinger equations using a model 2-point gluon function. The one phenomenological parameter and two current quark masses are fitted to reproduce $f_\pi$, $m_\pi$ and $m_K$. Our results for $f_K$, and for the vector mesons $\rho$, $\phi$, and $K^\star$ are in good agreement with the experimental values.

## 1. DYSON–SCHWINGER EQUATIONS

The Dyson–Schwinger equations [DSEs] form an excellent tool to study nonperturbative aspects of quark propagators and the formation of bound states [1]. The approach is consistent with quark and gluon confinement [2] and generates dynamical chiral symmetry breaking. It is straightforward to implement the correct one-loop renormalization group behavior of QCD [3], and obtain agreement with perturbation theory in the perturbative region. Phenomenological models based on the DSEs [4] have recently allowed significant progress not only for light mesons, but also for heavy mesons [5].

The DSE for the renormalized dressed-quark propagator in Euclidean formulation is

$$S(p)^{-1} = Z_2 \, i\gamma \cdot p + Z_4 \, m_q(\mu) + Z_1 \int^\Lambda \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma_\nu^a(q,p) \,, \quad (1)$$

where $D_{\mu\nu}(k)$ is the renormalized dressed-gluon propagator, $\Gamma_\nu^a(q;p)$ the renormalized dressed-quark-gluon vertex, and $\int^\Lambda$ denotes a translationally-invariant regularization of the integral with $\Lambda$ the regularization mass-scale. The general solution of Eq. (1) has the form $S(p)^{-1} = i\gamma \cdot p A(p^2) + B(p^2)$, renormalized according to

$$\left. S(p)^{-1} \right|_{p^2=\mu^2} = i\gamma \cdot p + m_q(\mu) \,, \quad (2)$$

with $m_q(\mu)$ the current quark mass at the renormalization scale $\mu$. The renormalization constants $Z_1$, $Z_2$ and $Z_4$ generally depend on $\mu$ and $\Lambda$, but not on the quark flavor.

The renormalized, homogeneous Bethe–Salpeter equation [BSE] for a bound state of a quark of flavor $a$ and an antiquark of flavor $b$ having total momentum $P$ is given by

$$\Gamma_M^{ab}(p; P) = \int^\Lambda \frac{d^4q}{(2\pi)^4} K(p,q;P) S^a(q+\eta P) \Gamma_M^{ab}(q; P) S^b(q - \bar\eta P) \,, \quad (3)$$

where $\eta + \bar\eta = 1$ describes momentum sharing, $\Gamma_M^{ab}(p; P)$ is the Bethe–Salpeter amplitude [BSA], and $M$ specifies the meson type: pseudoscalar, vector, axial-vector, or scalar. The

kernel $K$ operates in the direct product space of color and Dirac spin for the quark and antiquark and is the renormalized, amputated $\bar{q}q$ scattering kernel that is irreducible with respect to a pair of $\bar{q}q$ lines. This equation defines an eigenvalue problem with physical solutions at the mass-shell points $P^2 = -m^2$ with $m$ being the bound state mass.

We use a ladder truncation for the BSE

$$K_{tu}^{rs}(p, q; P) \rightarrow -\mathcal{G}((p-q)^2) \, D_{\mu\nu}^{\text{free}}(p-q) \left(\frac{\lambda^a}{2}\gamma_\mu\right)^{ru} \otimes \left(\frac{\lambda^a}{2}\gamma_\nu\right)^{ts}, \tag{4}$$

where $D_{\mu\nu}^{\text{free}}(k)$ is the bare gluon propagator in Landau gauge. Together with a rainbow truncation $\Gamma_\nu^a(q, p) \rightarrow \gamma_\nu \lambda^a/2$ for the quark DSE, Eq. (1), this preserves the axial-vector Ward–Takahashi identity [6], which ensures that in the chiral limit the ground state pseudoscalar mesons are massless even though the quark mass functions are strongly enhanced in the infrared. Thus pions are understood as both the Goldstone bosons associated with dynamical chiral symmetry breaking and as $\bar{q}q$ bound states [3,7].

For the "effective coupling" $\mathcal{G}(k^2)$ we employ the Ansatz [8]

$$\frac{\mathcal{G}(k^2)}{k^2} = \frac{4\pi^2}{\omega^6} D k^2 \mathrm{e}^{-k^2/\omega^2} + 4\pi \frac{\gamma_m \pi}{\frac{1}{2}\ln\left[\tau + \left(1 + k^2/\Lambda_{\text{QCD}}^2\right)^2\right]}\mathcal{F}(k^2), \tag{5}$$

with $\mathcal{F}(k^2) = [1 - \exp(-k^2/[4m_t^2])]/k^2$, $\tau = \mathrm{e}^2 - 1$, and $\gamma_m = 12/(33 - 2N_f)$. This Ansatz preserves the one-loop renormalization group behavior of QCD; in particular, one obtains the correct one-loop QCD anomalous dimension of the quark mass function $M(p^2) := B(p^2)/A(p^2)$ for both the chiral limit $(m(\mu) = 0)$ and the explicit chirally broken case $(m(\mu) \neq 0)$. The first term of Eq. (5) implements a strong infrared enhancement in the region $k^2 < 1$ GeV$^2$ which is phenomenologically required for sufficient dynamical chiral symmetry breaking and for confinement [9]. We use $m_t = 0.5$ GeV, $N_f = 4$, $\Lambda_{\text{QCD}}^{N_f=4} = 0.234$ GeV, $\omega = 0.4$ GeV, and a renormalization point $\mu = 19$ GeV.

## 2. MESON BETHE–SALPETER AMPLITUDES

The BSA $\Gamma_{PS}(q; P)$ of a pseudoscalar meson has the general form

$$\Gamma_{PS}(q; P) = \gamma_5 \left(iE + \gamma \cdot P \, F + \gamma \cdot q \, \hat{G} + \sigma_{\mu\nu} \, q_\mu P_\nu \, H\right), \tag{6}$$

with the invariant amplitudes $E$, $F$, $\hat{G}$ and $H$ being Lorentz scalar functions of $q^2$ and $q \cdot P$; $P^2 = -m_{PS}^2$ is the fixed, on-shell, meson momentum. For unflavored mesons that are charge conjugation eigenstates, such as the $\pi$, there is an additional constraint on the BSA to obtain a specified $C$-parity[1]. Of the four covariants given in Eq. (6), $\hat{G}$ is odd under $C$, the others are even under $C$. The only remaining quantity that can produce a desired uniform $C$-parity is $q \cdot P$ which is odd under $C$. Thus a pion will have an amplitude $\hat{G}$ that is odd in $q \cdot P$ while the other amplitudes are even in $q \cdot P$. Flavored mesons like kaons, will have amplitudes that are neither even nor odd in $q \cdot P$.

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[1]We do not discriminate between up and down quarks, nor incorporate electromagnetic corrections; therefore the BSA for $\pi^\pm$ is equal to that for $\pi^0$.

Massive vector mesons are transverse, and the general form of the BSA $\Gamma_\mu^V(q;P)$ can be expressed in terms of eight independent Lorentz covariants $T_\mu^i(q;P)$ with coefficients $F_i(q^2, q \cdot P; P^2)$. The choice for the covariants $T_\mu^i(q;P)$ to be used as a basis is constrained by the required properties under Lorentz and parity transformations, but is not unique; a convenient orthogonal basis is given in Ref. [8]. For the other mesons, scalar and axial-vector, covariants can be defined similarly. However, we expect the ladder truncation used here to be reliable for pseudoscalar and vector, but not for scalar mesons [6,10].

The electromagnetic decay mediated by a photon (e.g. $\rho^0$, $\omega$, $\phi$), and the leptonic decay mediated by a W-boson (e.g. $\rho^\pm$, $K^{\star\pm}$), are described by the vector decay constant [5,8,11]

$$f_V \, m_V \, \epsilon_\mu^{(\lambda)}(P) \;\; = \;\; \langle 0|\bar{q}^b \gamma_\mu q^a|V^{ab}(P,\lambda)\rangle \,, \tag{7}$$

where $\epsilon_\mu^{(\lambda)}$ is the polarization vector of the vector meson. This is analogous to

$$f_{PS} \, P_\mu \;\; = \;\; \langle 0|\bar{q}^b \gamma_\mu \gamma_5 q^a|PS^{ab}(P)\rangle \,,$$
$$\;\; = \;\; Z_2 \, N_c \int^\Lambda \frac{d^4q}{(2\pi)^4} \, \text{Tr}_D \left[ \gamma_\mu \gamma_5 S^a(q+\eta P)\Gamma_{PS}^{ab}(q;P)S^b(q-\bar{\eta}P) \right] \,, \tag{8}$$

for the pseudoscalar decay constant corresponding to $f_\pi = 131$ MeV under the normalization convention used here. The vector decay constant can be expressed as a loop integral

$$f_V \, m_V \;\; = \;\; \frac{Z_2 \, N_c}{3} \int^\Lambda \frac{d^4q}{(2\pi)^4} \, \text{Tr}_D \left[ \gamma_\mu S^a(q+\eta P)\Gamma_\mu^{ab}(q;P)S^b(q-\bar{\eta}P) \right] \,. \tag{9}$$

Both Eq. (8) and (9) are exact if the dressed quark propagators and the BSA are exact [5,7].

## 3. RESULTS FOR PSEUDOSCALAR AND VECTOR MESONS

After insertion of the general form of $\Gamma_{PS}(q;P)$ or $\Gamma_\mu^V(q;P)$ in terms of a covariant basis, the homogeneous BSE Eq. (3), can be reduced to a set of coupled integral equations for the scalar functions $F_i^{ab}(q^2, q \cdot P; P^2)$

$$F_i^{ab}(p^2, p \cdot P; P^2) \;\; = \;\; \int^\Lambda \frac{d^4q}{(2\pi)^4} \, \mathcal{K}_{i,j}^{ab}(p^2, p \cdot P; q^2, q \cdot P; p \cdot q; P^2)F_j^{ab}(q^2, q \cdot P; P^2) \tag{10}$$

This equation is solved numerically without further approximations as an eigenvalue problem to give the mesons masses; the decay constants follow from Eqs. (8) and (9).

The parameter $D$ and the current quark mass $m_{u/d}(\mu)$ are fixed by fitting $m_\pi$ and $f_\pi$. Next, the strange quark mass $m_s(\mu)$ is determined by a fit to $m_K$. With $D = 0.93$ GeV$^2$ and quark masses $m_{u/d}(\mu) = 5.5$ MeV, and $m_s(\mu) = 130$ MeV scaled to $\mu = 1$ GeV, we obtain a good description of the $\pi$, $\rho$, $K$, $K^\star$ and $\phi$ masses and decay constants, see Table 1. Our results are not very sensitive to the details of the gluon Ansatz [8], similar results are found with $\omega = 0.3$ GeV and $\omega = 0.5$ GeV, as long as $D$ and the quark masses are refitted. Also $r_\pi$ is in excellent agreement with the experimental value [12].

For small current quark masses, the pseudoscalar meson mass grows like $m_{PS}^2 \propto m_q(\mu)$, in agreement with the Gell-Mann–Oakes–Renner relation. For larger quark masses, $m_q(\mu) > 3\,m_s(\mu)$, the meson mass grows linearly with the current quark mass [13], as expected for heavy mesons. The decay constants also increase with increasing quark

Table 1

The pseudoscalar and vector meson masses $m_M$ and decay constants $f_M$ in GeV.

|         | $m$ (calc.) | $f$ (calc.) | $m$ (exp.) | $f$ (exp.) |
|---------|-------------|-------------|------------|------------|
| pion    | 0.138       | 0.131       | 0.1385     | 0.1307     |
| kaon    | 0.497       | 0.155       | 0.496      | 0.160      |
| $\rho$  | 0.742       | 0.207       | 0.770      | 0.216      |
| $K^\star$ | 0.936     | 0.241       | 0.892      | 0.225      |
| $\phi$  | 1.072       | 0.259       | 1.020      | 0.237      |

masses, as can be seen from Table 1. However, for heavy mesons it follows from the general form of the BSA, together with the normalization condition, that both the vector and the pseudoscalar decay constant decrease with the quark mass as $f_M \propto 1/\sqrt{m_M}$ [5].

For mesons that are charge conjugation eigenstates, the dependence of the amplitudes $F_i(q^2, q \cdot P; P^2)$ on $q \cdot P$ is minimal. For flavored mesons there is a much stronger angular dependence. Our results for physical observables, such as the mass and decay constant, are *independent* of the momentum sharing and of the regularization scale $\Lambda$, as long as all relevant covariants and the full angular dependence are included in the calculation.

**Acknowledgments**

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